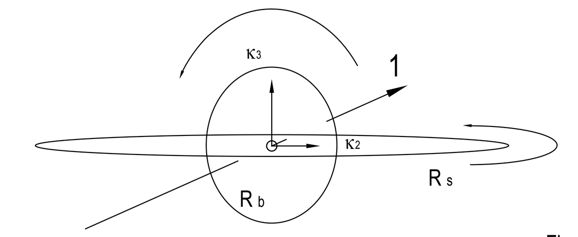
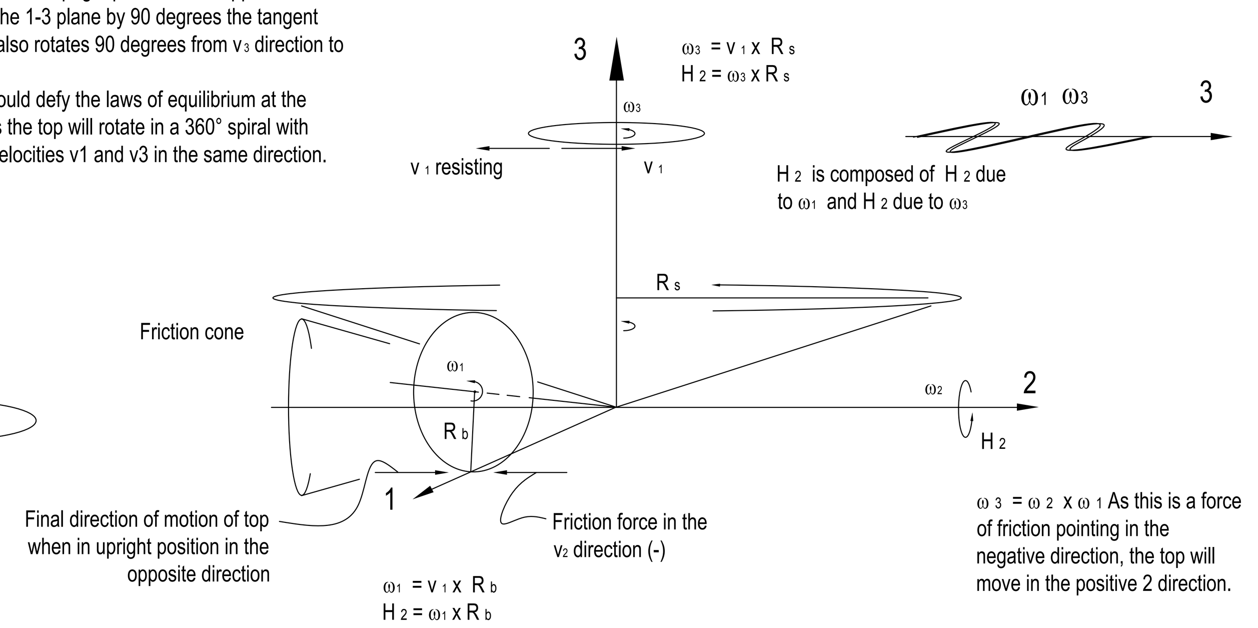
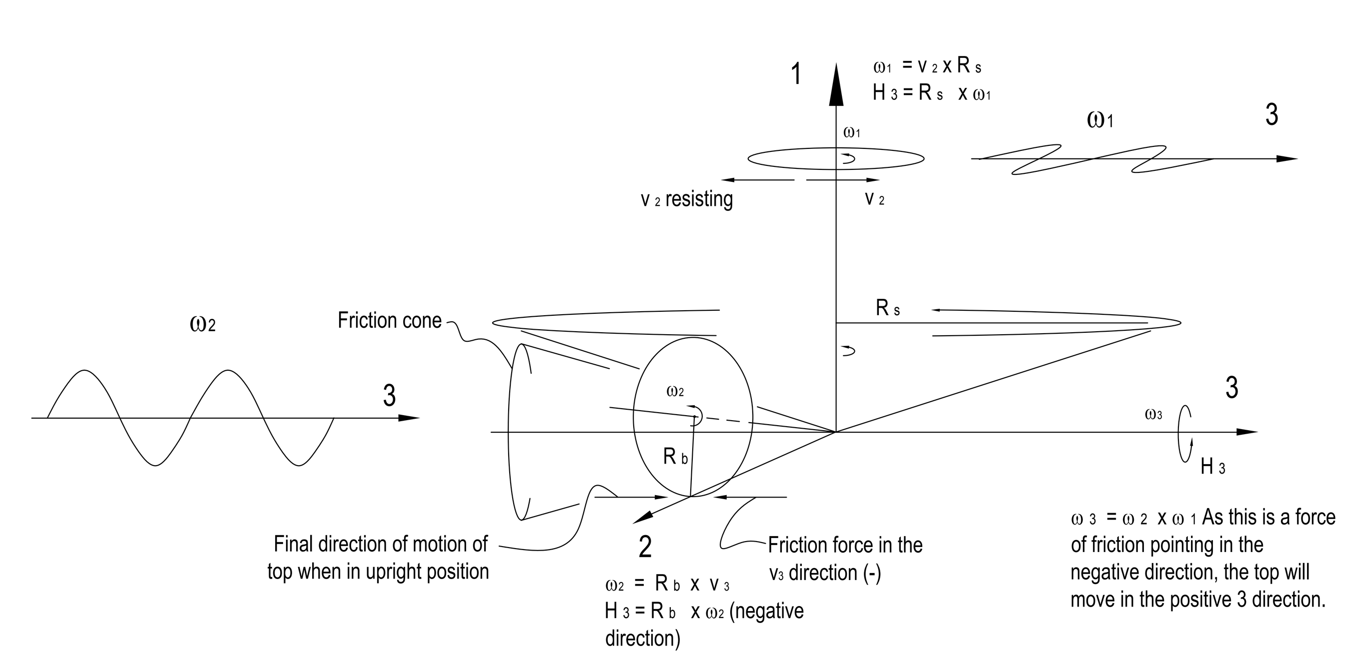
Elastic constants - The hyper cube and Riemann tensor

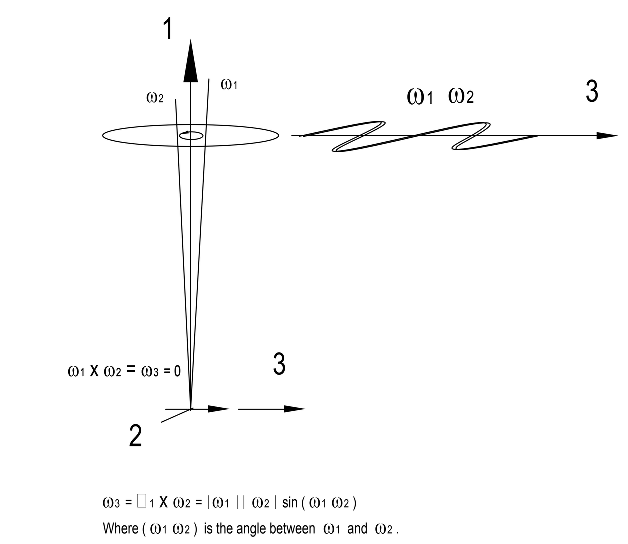
As the top rises to the upright position, and appears to have rotated in the 1-3 plane by 90 degrees the tangent velocity vector also rotates 90 degrees from v_3 direction to v_1 direction.
This however would defy the laws of equilibrium at the singularity. Thus the top will rotate in a 360° spiral with initial and final velocities v1 and v3 in the same direction.





The friction cone is actually what is referred to as the light cone composed of the space and body cones.



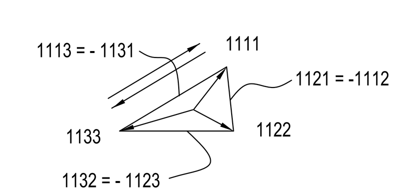


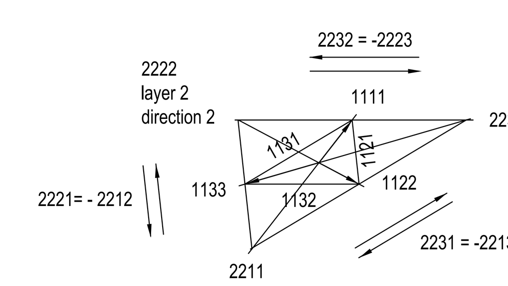
Rotational

Irrotational

Irrotational

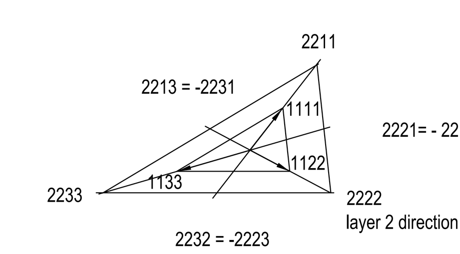
Clockwise rotation positive
(rotation is directly proportional to the edge length)





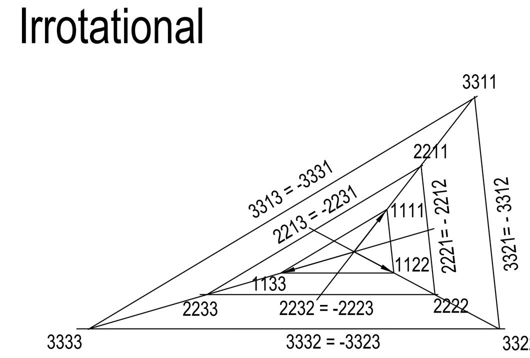
κ_{e1} = constant of proportionality for expansion layer 1
κ_{r1} = constant of proportionality for rotation layer 1

1113 = - 1131 μ $\kappa_{e1} \kappa_{r1}$. (2213 = -2231)
1121 = -1112 μ



κ_{e1} = constant of proportionality for expansion layer 1
$\kappa_{r1..6}$ = 1 = cos α

1113 = - 1131 μ κ_{e1} . (2213 = -2231)
1121 = -1112 μ κ_{e1} . (2221= - 2212)
1132 = - 1123 μ κ_{e1} . (2232 = -2223)



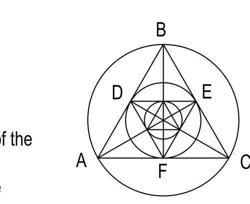
Interesting side note:
When we joint the midpoints of the sides of a triangle with a straight line, this line is equal to one half the third side.
DE = 0.5 AC
EF = 0.5 AB
DF = 0.5 BC
DE + EF + DF = 0.5 (AB + AC + BC)
The sum of the sides of the inner triangle is equal to one half the sum of the sides of the outer triangle.
If the sum of the sides of the outer triangle ABC is S, the sum of the sides of the inner triangle DEF = 0.5 S. The sum of the second inner triangle is 0.5^2 S and the sum of the third inner triangle is 0.5^3 S and we have a sequence whose first term is S and whose common ratio is 0.5 and whose number of terms may be n. if n is infinitely large, the sum of the sequence of the sums of the sides of the triangles is:

S - 0.5 ^n S / (1 - 0.5)

But if n is infinitely large, 0.5 ^n becomes very small and the sum of the sequence may be written as

S / (1 - 0.5) = 2 S

Hence the sum of the sides of all the triangles obtained by consecutively joining the midpoints of the sides of the inner triangles is equal to twice the sum of the sides of the first triangle.



Now that via rotation, we have eliminated one of the dimensions, we can show the compliance matrix in the 6x6 matrix as opposed to 9x9.

We have the usual 3x3 matrix, plus the 3 additional diagonal terms c44, c55, c66 which related to the extension in the 3 perpendicular cartesian axis due only to rotation.

Then c66 will be related to the top moving in the 3 direction when the rotation in this direction has approached zero.

As the vector perpendicular to the planes, 1-2, 1-3, 2-3, disappears so it does from the matrix. Cross the terms out shown in white, red and green.

Though not necessary, we can, re-arrange the matrix, which will enable us to see the terms that contribute to the extentions in the 11, 22, and 33 direction.

The terms 44, 55, 66, are friction/retardation terms due to rotation, which can be obtained by subracting the two rotations (with two components subtracted)

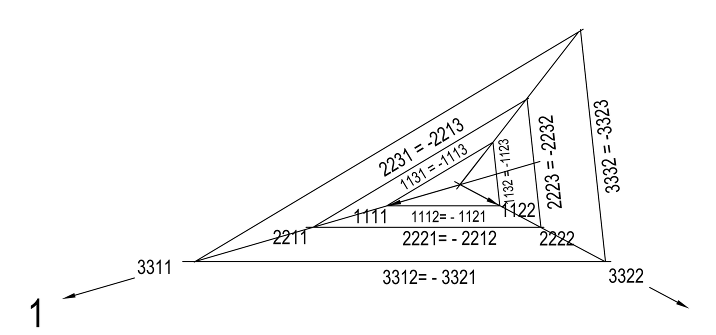
Components which indirectly contribute to the forces in the 3 direction would all be related to this retardation, and have to have rotation components as the cube say is squeezed in to a plane. If not, they are then eliminated.

In addition, if we have accounted for the rotation components perpendicular to say the 3-3 axis,then to avoid redundancy, components which contribute to this rotation and act on the remaining planes, 1-1, 2-2 , can be eliminated.

As the rotation vector perpendicular to the planes, 1-2, 1-3, 2-3, disappears so does its rotational components from the matrix. Cross the terms out shown in white, red and green.

$\varepsilon_{ij} = C_{ijlk} \sigma_{lk}$

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1112} & C_{1113} & C_{1121} & C_{1122} & C_{1123} & C_{1131} & C_{1132} & C_{1133} \\ C_{1211} & C_{1212} & C_{1213} & C_{1221} & C_{1222} & C_{1223} & C_{1231} & C_{1232} & C_{1233} \\ C_{1311} & C_{1312} & C_{1313} & C_{1321} & C_{1322} & C_{1323} & C_{1331} & C_{1332} & C_{1333} \\ C_{2111} & C_{2112} & C_{2113} & C_{2121} & C_{2122} & C_{2123} & C_{2131} & C_{2132} & C_{2133} \\ C_{2211} & C_{2212} & C_{2213} & C_{2221} & C_{2222} & C_{2223} & C_{2231} & C_{2232} & C_{2233} \\ C_{2311} & C_{2312} & C_{2313} & C_{2321} & C_{2322} & C_{2323} & C_{2331} & C_{2332} & C_{2333} \\ C_{3111} & C_{3112} & C_{3113} & C_{3121} & C_{3122} & C_{3123} & C_{3131} & C_{3132} & C_{3133} \\ C_{3211} & C_{3212} & C_{3213} & C_{3221} & C_{3222} & C_{3223} & C_{3231} & C_{3232} & C_{3233} \\ C_{3311} & C_{3312} & C_{3313} & C_{3321} & C_{3322} & C_{3323} & C_{3331} & C_{3332} & C_{3333} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{vmatrix}$$



The path of 3131 will take from level 3 to 1, (compress) then along diagonal 1113 (rotate) to point in the 11 direction with a negative sign if compression is taken as negative
The path of 3232 will take from level 3 to 2, then along diagonal 2232 to point in the 22 direction with a positive sign

C_{2121} The path of 2121 will take from level 2 to 1 (compress), then along diagonal 1121 (rotate) to point in the 11 direction, with a positive sign.

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1121} & C_{1122} & C_{1131} & C_{1132} & C_{1133} \\ C_{2111} & C_{2121} & C_{2122} & C_{2131} & C_{2132} & C_{2133} \\ C_{2211} & C_{2221} & C_{2222} & C_{2231} & C_{2232} & C_{2233} \\ C_{3111} & C_{3121} & C_{3122} & C_{3131} & C_{3132} & C_{3133} \\ C_{3211} & C_{3221} & C_{3222} & C_{3231} & C_{3232} & C_{3233} \\ C_{3311} & C_{3321} & C_{3322} & C_{3331} & C_{3332} & C_{3333} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{vmatrix}$$

Re-arrange row 2,3 and column 2,3

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1121} & C_{1131} & C_{1132} & C_{1133} \\ C_{2211} & C_{2222} & C_{2221} & C_{2231} & C_{2232} & C_{2233} \\ C_{2111} & C_{2122} & C_{2121} & C_{2131} & C_{2132} & C_{2133} \\ C_{3111} & C_{3122} & C_{3121} & C_{3131} & C_{3132} & C_{3133} \\ C_{3211} & C_{3222} & C_{3221} & C_{3231} & C_{3232} & C_{3233} \\ C_{3311} & C_{3322} & C_{3321} & C_{3331} & C_{3332} & C_{3333} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{21} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{vmatrix}$$

Re-arrange row 3,6 and column 3,6

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1131} & C_{1132} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2231} & C_{2232} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3331} & C_{3332} & C_{3321} \\ C_{3111} & C_{3122} & C_{3133} & C_{3131} & C_{3132} & C_{3121} \\ C_{3211} & C_{3222} & C_{3233} & C_{3231} & C_{3232} & C_{3221} \\ C_{2111} & C_{2122} & C_{2133} & C_{2131} & C_{2132} & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

Tensor notation	11	22	33	32, 23	31, 13	12, 21
Matrix notation	1	2	3	4	5	6

See Nye, physical properties of crystals, pg. 134

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} \\ C_{2211} & C_{2222} & C_{2233} \\ C_{3311} & C_{3322} & C_{3333} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{vmatrix}$$

** Higher order terms are ignored

C66 is then the Rotation / Friction / Retardation component perpendicular to the 21 plane in the 3-3 direction $C_{2121} = C_{66}$

C44 is then the Rotation / Friction / Retardation component perpendicular to the 32 plane in the 1-1 direction $C_{3232} = C_{44}$

C55 is then the Rotation / Friction / Retardation component perpendicular to the 31 plane in the 2-2 direction $C_{3131} = C_{55}$

Finally , Re-arrange row 4,5 and column 4,5

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1132} & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2232} & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3332} & C_{3331} & C_{3321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3232} & C_{3231} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3132} & C_{3131} & C_{3121} \\ C_{2111} & C_{2122} & C_{2133} & C_{2132} & C_{2131} & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

C44 is then the Rotation / Friction / Retardation component perpendicular to the 32 plane in the 1-1 direction $C_{3232} = C_{44}$

C55 is then the Rotation / Friction / Retardation component perpendicular to the 31 plane in the 2-2 direction $C_{3131} = C_{55}$

C66 is then the Rotation / Friction / Retardation component perpendicular to the 21 plane in the 3-3 direction $C_{2121} = C_{66}$

Quickly eliminate the following, their last indecies have been accounted for in the main diagonals

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1132} & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2232} & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3332} & C_{3331} & C_{3321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3232} & & \\ C_{3111} & C_{3122} & C_{3133} & & C_{3131} & \\ C_{2111} & C_{2122} & C_{2133} & & & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

Keep only the main rotational components

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & & & \\ C_{2211} & C_{2222} & C_{2233} & & & \\ C_{3311} & C_{3322} & C_{3333} & & & \\ & & & C_{3232} & & \\ & & & & C_{3131} & \\ & & & & & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1132} & & \\ C_{2211} & C_{2222} & C_{2233} & & C_{2231} & \\ C_{3311} & C_{3322} & C_{3333} & & & C_{3321} \\ C_{3211} & & & C_{3232} & & \\ & C_{3122} & & & C_{3131} & \\ & & C_{2133} & & & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

ε_{32} is acting in the 1 direction so leave the components which act in the 11 direction, eliminate the rest.

ε_{31} is acting in the 2 direction so leave the components which act in the 22 direction, eliminate the rest.

ε_{21} is acting in the 3 direction so leave the components which act in the 33 direction, eliminate the rest.

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2232} & & \\ C_{3311} & C_{3322} & C_{3333} & & & \\ & C_{3222} & & C_{3232} & & \\ C_{3111} & & & & C_{3131} & \\ C_{2111} & & & & & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

ε_{32} is acting in the 2 direction so leave the components which act in the 22 direction, eliminate the rest.

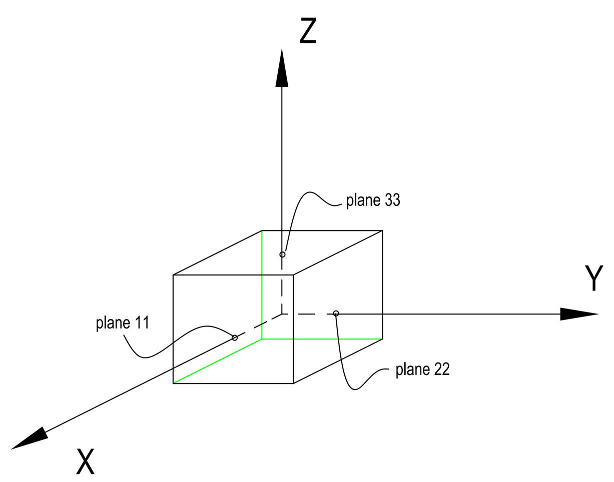
ε_{31} is acting in the 1 direction so leave the components which act in the 11 direction, eliminate the rest.

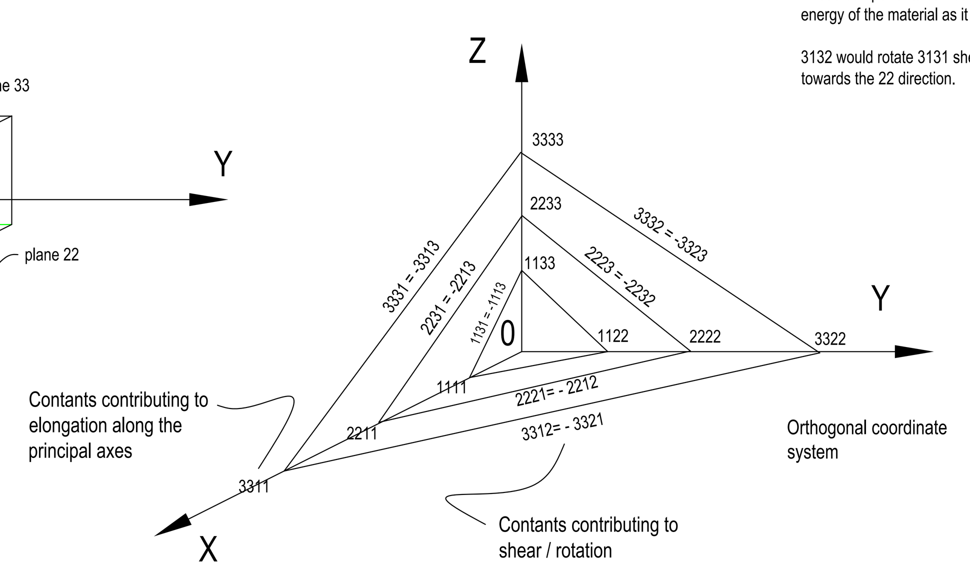
ε_{21} is acting in the 1 direction so leave the components which act in the 11 direction, eliminate the rest.

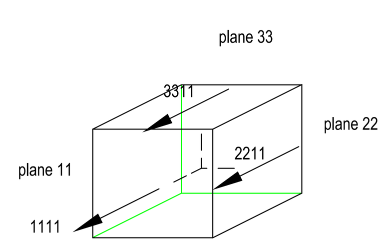
$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1132} & & \\ C_{2211} & C_{2222} & C_{2233} & & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & & C_{3331} & C_{3321} \\ C_{3211} & & & C_{3232} & & \\ & C_{3122} & C_{3133} & & C_{3131} & \\ & C_{2122} & C_{2133} & & & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{vmatrix}$$

Rotation components
Rotation components

Elastic constants - The Hypercube and Riemann tensor - Continued





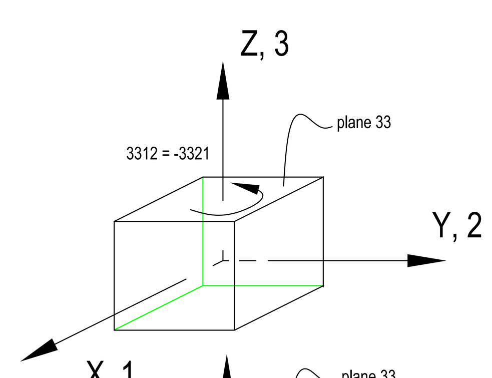


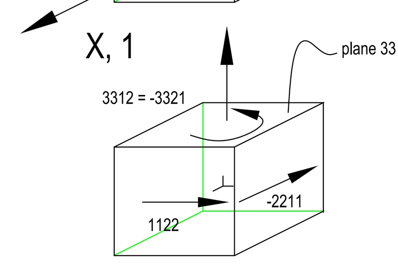
Now take a cube and the plane 11 perpendicular to the X or 11 axis.
the stress 1111 is a stress which acts on the plane perpendicular to the 11 axis and is parallel in direction to the 11 axis.
Similarly, the stress 3311 acts on the plane perpendicular to the 33 or z axis and is paralel in direction to the x or 11 axis, and so forth.

The component 3312 acts on the 33 plane trying to rotate the upper face of the cube in a direction from 1 to 2 (y to x). Counter clockwise rotation is considered positive.

it is then obvious that 3312 is dependant on 2211 and 1122 which also tend to rotate the cube itself by acting on faces 22 and 11 in the positive 2 direction and negative one direction, respectively.

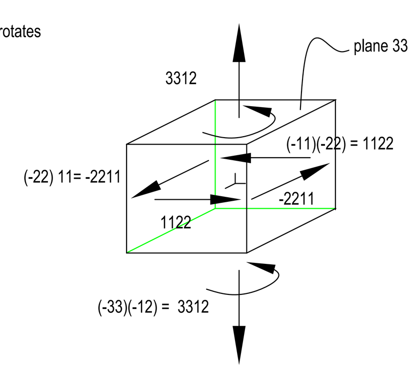
Hence stresses 1122 and -2211 (minus sign because 2211 acts in the negative x direction) affect the cube in a similar fashion.





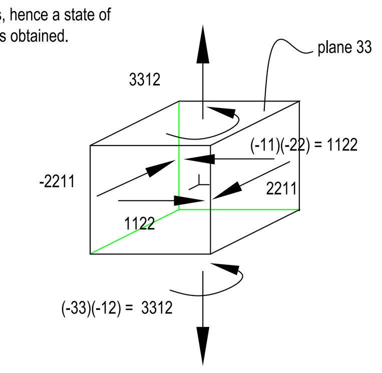
Now for there to be equilibrium, we have to show all the forces, which means the rotation -3321 which cancels 3312 and the other two shear forces on the negative faces first and then reverse the direction of the shears so as to negate this rotation.

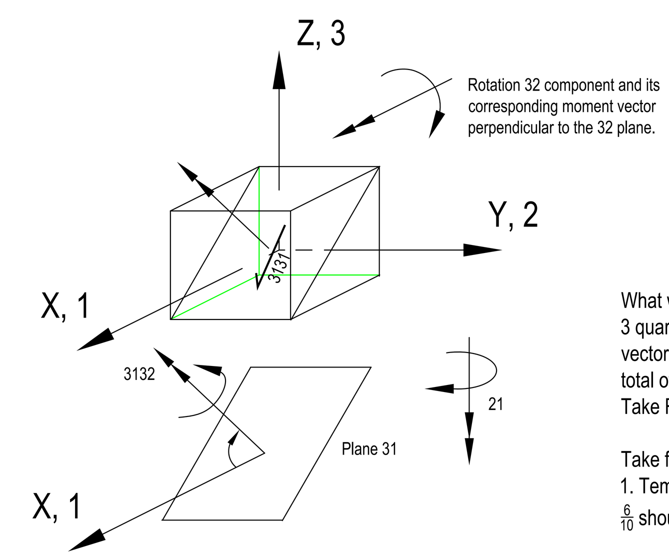
Cube rotates



We can shange the direction of rotation on the bottom face of the cube to maintain equilibrium. Hence for equilibrium we must have 3312 = -3312 if the shears act as they are shown in the figure to the left.

Cube is prevented from rotation by changing the direction of the shear forces, hence a state of equilibrium is obtained.

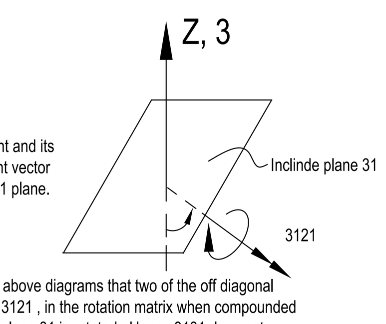




This rotation component becomes fully effective when the plane is inclined and is equal to zero when the plane is horizontal.

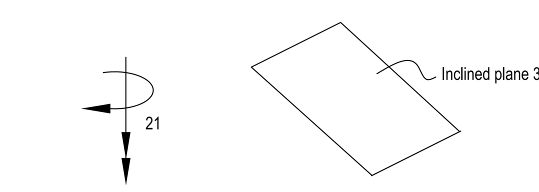
These components can be attributed to the energy of the material as it breaks appart.

3132 would rotate 3131 shear component towards the 22 direction.

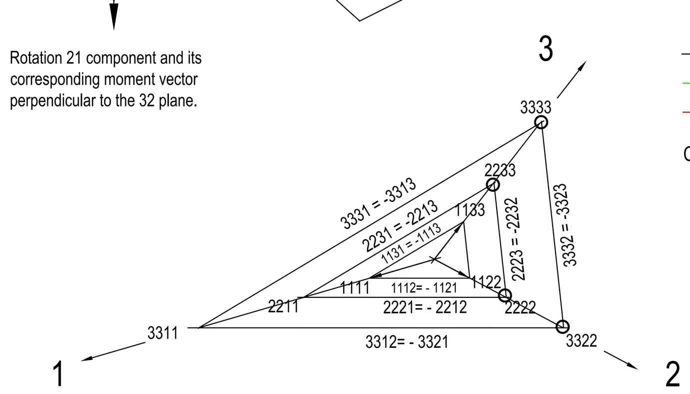


Rotation 21 component and its corresponding moment vector perpendicular to the 31 plane.

We see from the above diagrams that two of the off diagonal terms, 3132 and 3121 , in the rotation matrix when compounded cancel when the plane 31 is rotated. Hence 3131 does not rotate.



Rotation 21 component and its corresponding moment vector perpendicular to the 32 plane.



For the component 1132:
If we use the level change and rotation method, then the six components shown below should equal to zero. For 1132, 11 stays in level one axis 1, and 32 is not a path to the 2 direction.
But the plane 11 is not inclined, hence we can have a rotation vector 32 perpendicular to it.

But 1131 would be a rotation which would show up as the plane starts to tilt due to the applied loads.

From the two observations above we see that 1131 should be zero initially.

These are the two methods of determining whether the constants are zero or not.

Compliance constants with the color magenta have a tendancy to rotate the plane about the axis which is perpendicular to the normal to the plane.

Compliance constants with the color green have a tendancy to rotate the plane about the axis of the normal.

Compliance constants with the color red tend to rotate the inclined plane about the straight/upright normal.

Rotation matrix

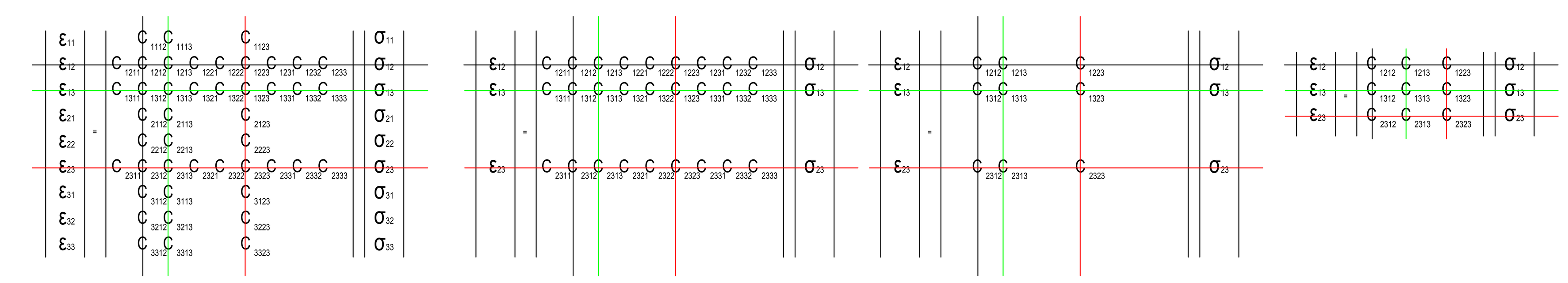
$$\begin{vmatrix} \varepsilon_{31} \\ \varepsilon_{32} \\ \varepsilon_{21} \end{vmatrix} \begin{vmatrix} C_{3131} & C_{3132} & C_{3121} \\ C_{3231} & C_{3232} & C_{3221} \\ C_{2131} & C_{2132} & C_{2121} \end{vmatrix} \begin{vmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{21} \end{vmatrix}$$

Rotation components

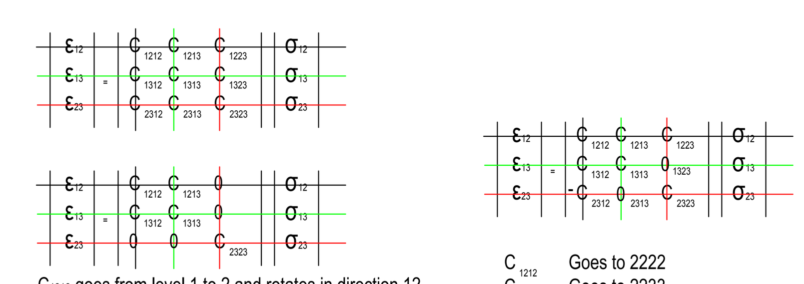
** Higher order terms are ignored

What was eliminated, which started it all!
3 quarks (rotation vectors which we eliminated to get the 6x6 matrix) each of which gives rise to two vectors . By eliminating each vector we have two vectors remaining each with 3 components for a total of 6.
Take Radiation as the 10th column in the 9x9 matrix.

Take for the composition the inverse ratio in decreasing order of magnitude:
1. Temperature (heat / radiation) 3. Aether (rotation matrix) 6. Matter (stress /strain matrix)
$\frac{6}{10}$ should be radiation $\frac{3}{10}$ should be Aether $\frac{1}{10}$ should be matter



Invert and apply as friction tensor.

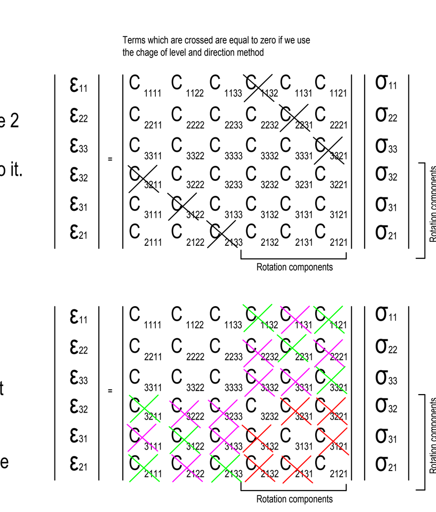


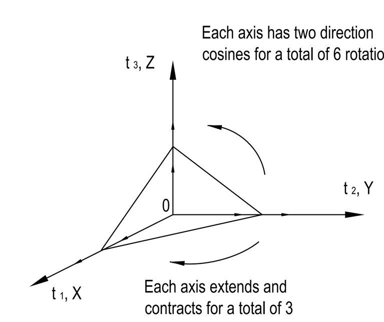
C_{1212} goes from level 1 to 2 and rotates in direction 12

C_{1212}	Goes to 2222
C_{1213}	Goes to 2233
C_{1223}	Zero
C_{1312}	Goes to 3322
C_{1313}	Goes to 3333
C_{1323}	zero
C_{2312}	Zero
C_{2313}	zero
C_{2323}	Goes to 3333

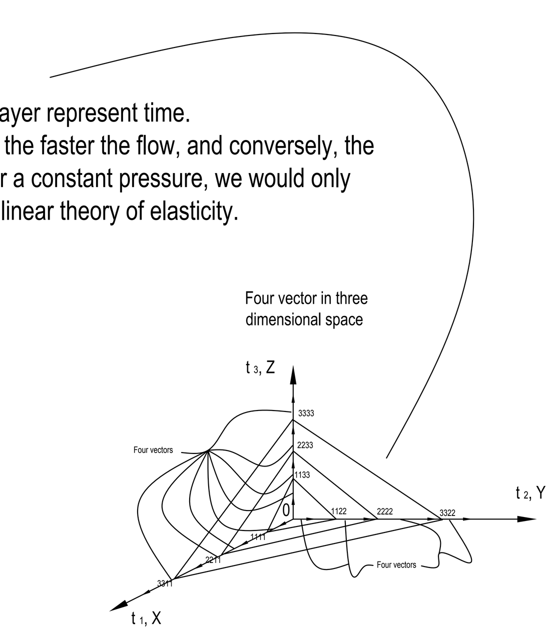
C_{1212}	Goes to 2222
C_{1213}	Goes to 2233
C_{1223}	Goes to 2233
C_{1312}	Goes to 3322
C_{1313}	Goes to 3333
C_{1323}	Zero
C_{2312}	Negative goes to 3311
C_{2313}	Zero
C_{2323}	Goes to 3333

Terms which are crossed are equal to zero if we use the chage of level and direction method





Rate of flow vs curvature of the surface
Let distance between the surfaces in each layer represent time.
The closer the surfaces are to one another the faster the flow, and conversely, the farther the surfaces the slower the flow. For a constant pressure, we would only have one layer, called the dihedral plane in linear theory of elasticity.



Visualizing the number of constants in the Riemann Tensor:

Take the first plane: there are three lengths on the x, y, and z axis which can be varried, depending on the orientation of the plane (3). Each of the axis rotates towards the others for a total of 6 possible movements. (each axis is skewed from the other two by the two direction cosines for a total of 6 direction cosines at each layer for three layers this amounts to 18 constants) . So now we have a total of 6+3=9 constants. Now put a vector on the point located on the plane. The resultant would have 5 constants, 3 for magnitude and 2 for direction. Total constants for the first plane would amount to 9+5 = 14.

Take two planes: there are 3 x 2 = 6 for the lengths and 2 x 6 =12 for a total of 18 constants.
Having obtained the resultant , add 5 for its 3 components and two directions and we have 23 constants.

Counting the vector from the third plane outwards:
As before, we have 3 x 3 = 9 for the lengths and 3 x 6 = 18 constants. The total now is 27 constants.
Add 5 more for the resultant and we have 32 constants.
Now for a cube we have 8 such vertices so 8 x 32 = 256 total. These are the constants on the Riemann sphere.
The uniform expansion-no rotation: gives us 9 constants for the lengths plus 5 = 14.

In other words, we have the 27 + 2 = 29 with the two rotations. Define the point in space by 3 more constants and for a vector on a plane passing through this point we have a total of 32 constants.
For the 27 constants, take non-uniform expansion and prevent rotation about one axis.
we had 18 constants (6 x 3 layers) = 18

Factor in the non-uniform expansion of the 3 vectors for 3 layers for a total of 9 additional constants. This will amout to 18 + 9 = 27 constants. Prevent rotation about one axis for a total of 29 constants.
A point is specified in space by three constants. Adding these to the 29 constants results in 32 constants total.

For a hyper cube we have eight such vertices and hence 8 x 32 = 256 constants.

For two dimensions we have two lengths and one angle for a total of 3 constants. 3 x 3 planes = 9 constants. The resultant has a given magnitude and direction for an additional two which totals 11 constants. For four quadrants this amounts to a total of 44 constants.

Since one can derive the direction cosines given the component of each vector which lies on the plane, the problem reduces to a 9x9 matrix with 81 components. This is in effect the same as saying we have merged the three planes into one. In other words, there are three normals to the deviator plane which can be averaged to represent the mean normal. This normal will then deviate by what is called the change of phase of similitude of deviators, ie; from the resultant.